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Generalized formulation for strip yielding model with variable cohesion and its analytical solutions

Wei Wang [a], Cheng-Tzu Thomas Hsu [b,*], Denis Blackmore [c]

[a] HLW International, LLP. 115 5th Avenue New York, NY 10003, USA
[b] Department of Civil and Environmental Engineering, New Jersey Institute of Technology, University Heights, Newark, NJ 07102-1982, USA
[c] Department of Mathematical Sciences, New Jersey Institute of Technology, University Heights, Newark, NJ 07102-1982, USA

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Abstract

A generalized plastic zone theory based on the Paris' displacement formula is presented to study inelastic fracture properties. This theory is capable of analyzing inelastic fracture characteristics of engineering materials in general. The so-called size effect is predicted for softening materials. A brittleness index is also discussed based on this theory. It is illustrated that the brittleness index may be used to characterize the inelastic fracture properties of the structures. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Formulation; Strip yielding model; Cohesion; Analytical solutions

1. Introduction

The plastic zone size may be estimated by Irwin's (1960) effective crack length method for metallic structures. This effective crack length concept is based on the fact that most metallic materials are considerably ductile and have a nearly constant yield strength after the stress in the material reaches its peak value. His plastic zone formula is based on the concept of force balance near the crack tip and is considerably accurate in small scale yielding cases when compared with the more accurate Dugdale's strip yielding model. However, extension of this theory to cementitious material, such as mortar and concrete may initiate technical problems due to the softening properties of the material.

Concrete and mortar are known to be softening materials and have a descending cohesive force. Previous researches have shown differences in the fracture properties between cementitious materials and metals (Carpinteri, 1985; Bazant, 1985; Shah, 1984; Sih, 1984). Several researchers concluded that this softening cohesive force plays a key role in determining the fracture properties of the concrete structures, especially in the case of non-linear fracture mechanics.

* Corresponding author.

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The so-called fictitious crack model proposed by Hillerborg (1976, 1979, 1985) has been widely accepted as an effective method in modeling the fracture properties of concrete and reinforced concrete structures. This model is based on the classical plastic zone theory by applying variable cohesive force to the plastic zone. However, this concept has been discussed among researchers in the field of fracture of metallic structures. The use of variable cohesion in the plastic zone to model the hardening properties of different metals was proposed by Chen (1975). The application of this method, however, was limited due to mathematical difficulties, especially for the materials with variable cohesion. Chen realized this and suggested the use of numerical method to solve individual problems (Chen, 1975).

The development of digital computer technology and numerical methods, especially the finite element method, makes the fictitious crack model viable. This model, originally proposed by Hillerborg revealed a high potential in predicting the fracture properties of concrete structures. Numerous researches and technical papers have been published in the literature on this topic since the 1970s. However, most of these researches concentrated on the analysis methods and case studies.

The development of an analytical method may extend the use of plastic zone theory to predict the fracture and failure and fatigue properties of softening materials in engineering practice. The key issues are (1) Prediction of the length of the plastic zone in a cementitious material, and (2) The crack opening displacement in the plastic zone.

These two problems are fundamental to the application of fracture mechanics to cementitious materials. In the following sections of this article, a generalized plastic zone theory is presented based on the Paris' displacement formula. A closed form solution derived from the governing integral equations of the generalized plastic zone theory using functional analysis (Wang, 1995) will be illustrated.

The study of plastic zone formation and propagation leads to a better understanding of fatigue crack growth. A well formulated closed form solution of plastic zone size, cracking opening shape, and crack tip opening size will also provide much needed tools in qualifying energy release rate for energy based crack growth rate theory (Wang, 1991; Wang and Hsu, 1994).

2. Generalized plastic zone theory

Paris (1957), Paris and Erdogan (1963) and Taha et al. (1973) proposed a method to calculate certain displacements relevant to crack problems. His method was based on Castigliano's theorem and fracture mechanics. Assume that the total strain energy of a cracked body is U under an external load of P. The crack opening displacement Δ_F (Fig. 1) was found as

$$\Delta_F = \frac{2}{E'} \int_{a_F}^{a_c} K_{IP} \frac{\partial K_{IF}}{\partial F} \, da, \tag{1}$$

where K_{IF} is the stress intensity factor caused by a couple of virtual forces F on the position in question; K_{IP}, the stress intensity factor corresponding to the external load P; a, the integral variable; and a_c and a_F are the position of the crack tip and the position where the displacement is to be calculated.

Using the principle of superposition, the Paris formula can be applied to the classical plastic zone theory and leads to a generalized approach to the problem. Note that the stress intensity factor of a linear elastic cracked body shall be a linear function of the applied load. Thus $\partial K_{IF}/\partial F$ in Paris' formula can be considered as the virtual strength intensity factor, $k^D(\xi, x)$, corresponding to a unit force F. Paris' formula can be written as:

$$\delta(x) = \int_0^a K(\xi) k^D(\xi, x) \, d\xi, \tag{2}$$

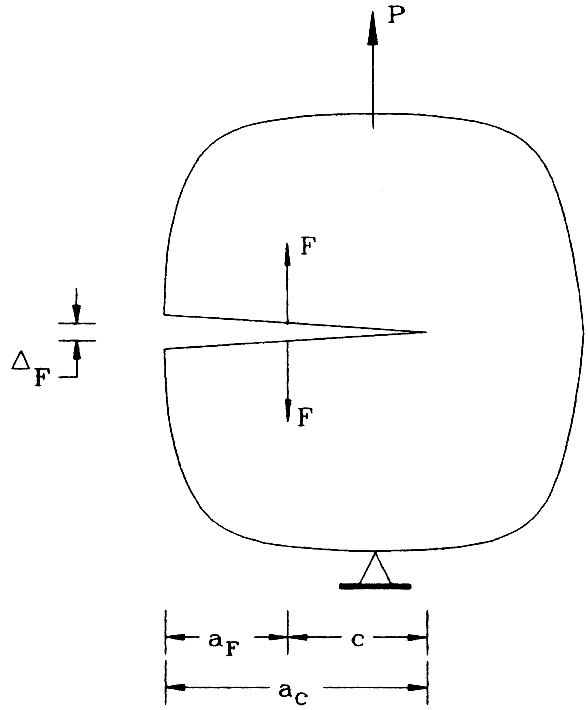

Fig. 1. Crack opening displacement by Castigliano's theorem.

where a is the crack length, K, the stress intensity factor, and ξ, the moving coordinate along the crack length.

In the case of a distributed cohesive force $\sigma(b)$ exists in the plastic zone $[a, c]$, (b varies between a and c), the negative contribution by the distributed cohesive force to the total stress intensity factor shall be considered in computing the crack opening displacement in Eq. (2) . By using the principle of superposition, the crack opening displacement can be found as (Fig. 2):

$$\delta(x) = \delta_1(x) + \delta_2(x), \tag{3}$$

where

$$\begin{aligned}\delta_1(x) &= \frac{2}{E'}\int_x^c \left(\frac{K(\xi)}{2} - \int_a^x K_c(\xi,\delta,b)\,\mathrm{d}b\right) k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi \\ &= \frac{2}{E'}\int_x^c \frac{K(\xi)}{2} k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi - \frac{2}{E'}\int_x^c \left(\int_a^\xi K_c(\xi,\delta,b)\,\mathrm{d}b\right) k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi,\end{aligned} \tag{4}$$

$$\delta_2(x) = \frac{2}{E'}\int_x^c \frac{K(\xi)}{2} k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi - \frac{2}{E'}\int_x^c \left(\int_a^\xi K_c(\xi,\delta,b)\,\mathrm{d}b\right) k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi. \tag{5}$$

Eqs. (4) and (5) lead to a singular integral equation:

$$\delta(x) = \frac{2}{E'}\int_x^c K(\xi) k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi - \frac{2}{E'}\int_x^c k^{\mathrm{D}}(\xi,x)\,\mathrm{d}\xi \int_a^\xi K_c(\xi,\delta,b)\,\mathrm{d}b. \tag{6}$$

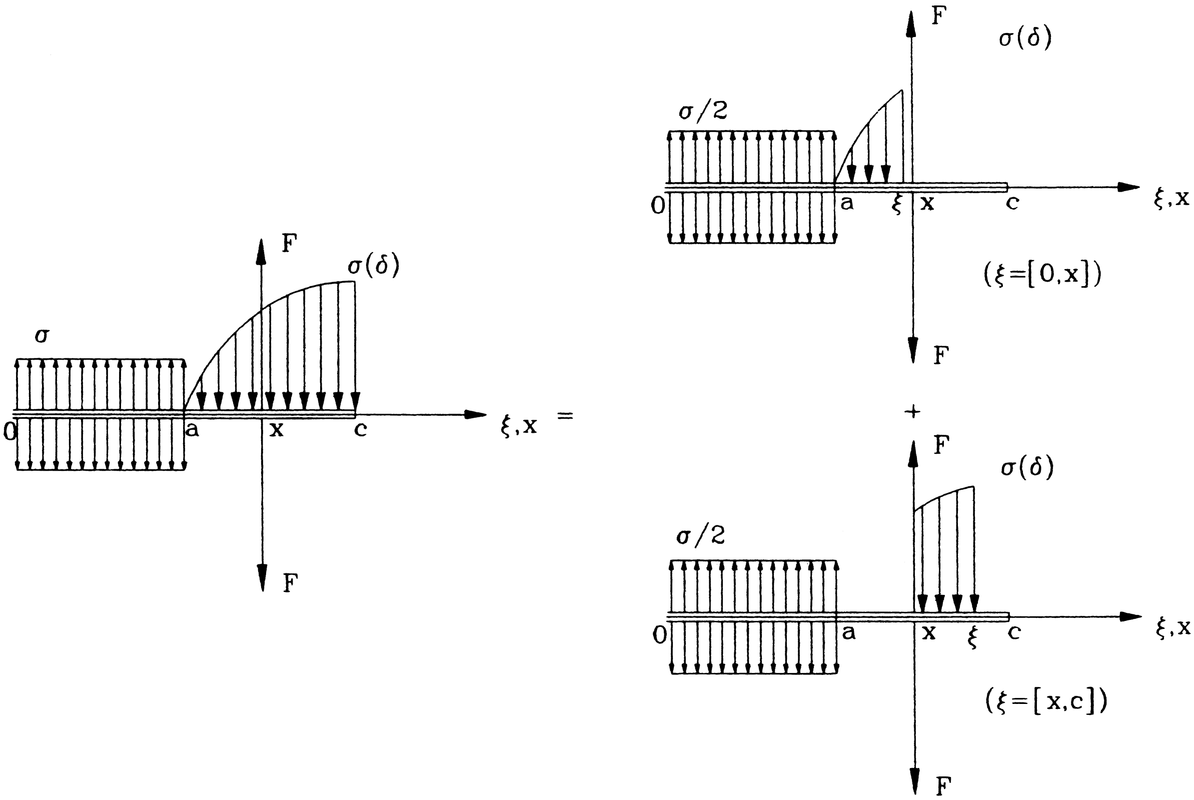

Fig. 2. Generalized process theory by Paris' displacement formula.

The smooth closure condition can be used to identify the boundary condition needed to solve the problem. The physical meaning of the smooth closure condition is that the stress intensity factor at the tip of a plastic zone is equal to zero, which indicates that the stress field has no singularity at this point, (Dugdale, 1960), i.e.,

$$K(c) - \int_a^c K_c(\xi, \delta, b)\,\mathrm{d}b = 0. \tag{7}$$

As an example, the solution for a wide plate with a central crack problem (constant distributed cohesive force) is illustrated subsequently. In the case of variable distributed cohesive forces, Eq. (6) becomes a singular integral equation with a free boundary condition as specified in Eq. (7). Closed form solutions for the above equation do not exist in general.

However, when the material has a constant cohesion, Eqs. (6) and (7) become an integration of a given function, and thus, may have exact closed form solutions. In fact, most ductile metals can be considered to have a constant cohesion after yielding. Hence, the applications of Eqs. (6) and (7) to metallic structures have led to good results (Rice, 1966, 1967, 1968).

As an example, the exact solution for the problem of a wide plate with a single central crack (see Fig. 3) is derived to illustrate the proposed generalized plastic zone theory.

The stress intensity factor of the specimen under a distant uniform stress of such a specimen is given below:

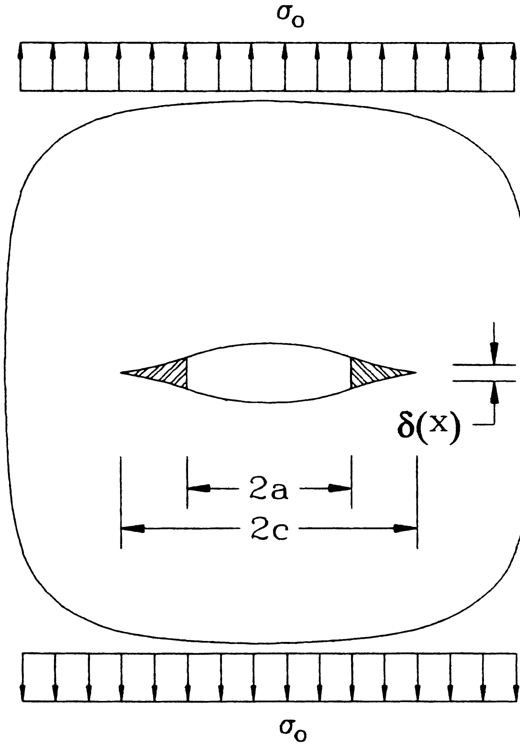

Fig. 3. Centrally cracked wide plate subjected to distant uniform tensile stress.

$$K(\xi) = \sqrt{\pi\xi}\sigma_0. \tag{8}$$

The virtual stress intensity factor is

$$k^{\mathrm{D}}(\xi, x) = \frac{2}{\sqrt{\pi\xi}} \frac{\xi}{\sqrt{\xi^2 - x^2}}, \tag{9}$$

and the stress intensity factor by the cohesive force in a unit length is

$$K_{\mathrm{c}}(\xi, \delta, b) = \frac{2}{\sqrt{\pi\xi}} \frac{\xi\sigma_y}{\sqrt{\xi^2 - b^2}}. \tag{10}$$

Thus, the crack opening displacement $\delta(x)$ can be found by Eq. (6)

$$\delta(x) = \frac{4\sigma_0}{E'}\sqrt{c^2 - x^2} - \frac{8}{\pi E'}\int_x^c \frac{\xi}{\sqrt{\xi^2 - x^2}}\,\mathrm{d}\xi \int_a^{\xi} \frac{\sigma_y}{\sqrt{\xi^2 - b^2}}\,\mathrm{d}b, \tag{11}$$

and the boundary condition reads:

$$\sqrt{\pi c}\sigma_0 = \int_a^c \frac{2}{\sqrt{\pi c}} \frac{c\sigma_y}{\sqrt{c^2 - b^2}}\,\mathrm{d}b \tag{12}$$

or

$$\frac{\pi\sigma_0}{2} = \int_a^c \frac{\sigma_y}{\sqrt{c^2 - b^2}}\,\mathrm{d}b \tag{13}$$

A formula similar to Eq. (11) was proposed by Chen (1975) by using Paris' formula for this particular problem. However, the boundary condition used in Chen's research was derived by vanishing of an integral Kernel of the equation. For the problem with constant cohesion, his research led to the same result listed hereafter.

Eqs. (10) and (13) lead to the following solutions:

$$\delta(x) = \frac{4}{E'}\sqrt{c^2-b^2}\left(\sigma_0 - \frac{2\sigma_y}{\pi}\cos^{-1}\frac{a}{c}\right) + \frac{4\sigma_y}{\pi E'}\left[a\ln\left(\frac{\sqrt{c^2-a^2}+\sqrt{c^2-x^2}}{\sqrt{c^2-a^2}-\sqrt{c^2-x^2}}\right)\right.$$
$$\left. - x\ln\left(\frac{x\sqrt{c^2-a^2}+a\sqrt{c^2-x^2}}{x\sqrt{c^2-a^2}-a\sqrt{c^2-x^2}}\right)\right], \tag{14}$$

and

$$\sigma_0 = \frac{2\sigma_y}{\pi}\cos^{-1}\frac{a}{c} \tag{15}$$

Eq. (14) can be simplified by using Eq. (15),

$$\delta(x) = \frac{4\sigma_y}{\pi E'}\left[a\ln\left(\frac{\sqrt{c^2-a^2}+\sqrt{c^2-x^2}}{\sqrt{c^2-a^2}-\sqrt{c^2-x^2}}\right) - x\ln\left(\frac{x\sqrt{c^2-a^2}+a\sqrt{c^2-x^2}}{x\sqrt{c^2-a^2}-a\sqrt{c^2-x^2}}\right)\right]. \tag{16}$$

The above solutions are identical to those obtained by using the Westgård stress function (Dugdale, 1960). This solution has been widely used to verify different approximations for non-linear fracture mechanics.

3. Non-linear fracture characteristics of softening materials

Eq. (16) shows the crack opening displacement in the plastic zone of materials with perfectly plastic cohesion, which is a good approximation for metal structures. However, as mentioned in the previous sections, concrete is a typical softening material with a descending cohesive force. Thus, this solution is not applicable to the problems of this kind; instead, a variable cohesion shall be used. This variable cohesion is known to be a function of crack opening displacement for cementitious material. Thus, Eq. (6) becomes a singular integral equation with a free boundary problem. Its boundary condition is given by Eq. (7).

General solutions for integral equations of this kind are not available. A detailed solution for a variable cohesive force is illustrated in the following section using an iterative functional analysis method. This method is expected to have a fast convergence within the range of the practical problems. The numerical experiment presented in the following sections illustrates the fast convergence as expected from the analysis.

In case of a varying cohesion $s[d]$, the governing equation for a centrally cracked plate is

$$\delta(x) = \frac{4\sigma_0}{E'}\sqrt{c^2-x^2} - \frac{8}{\pi E'}\int_x^c \frac{\xi}{\sqrt{\xi^2-x^2}}\,\mathrm{d}\xi\int_a^\xi \frac{\sigma[\delta(b)]}{\sqrt{\xi^2-b^2}}\,\mathrm{d}b \tag{17}$$

and the boundary condition is

$$\frac{\pi\sigma_0}{2} = \int_a^c \frac{\sigma[\delta(b)]}{\sqrt{c^2-b^2}}\,\mathrm{d}b \tag{18}$$

For simplicity, a linear cohesion vs. crack opening displacement relation is assumed in the analysis (see Fig. 4). The use of a linear curve to approximate the descending part of soften material was proposed by Hillerborg et al. (1976) to study the non-linear fracture properties of concrete. In the past, different models

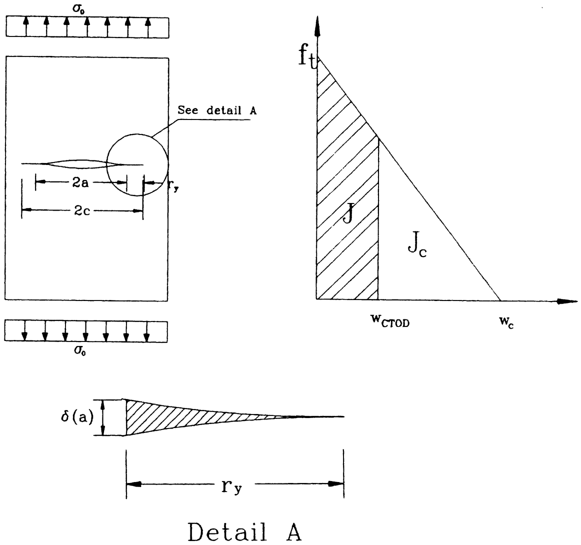

Fig. 4. Centrally cracked wide plate subjected to distant tensile stress (softening materials).

were proposed in numerous studies, they varied from simple linear curves to exponential curves. Also, they all achieved certain objectives in modeling the fracture behavior of softening materials, such as concrete and mortar (Shah et al., 1995). Eqs. (17) and (18) are similar to Eqs. (11) and (13). However, the constant cohesion is replaced by a function of $\sigma[\delta(x)]$. Thus, Eq. (17) becomes a non-linear Volterra integral equation, except for the fact that the limit c also depends on δ. General solutions of the integral equations of this kind are not available.

Picard iteration (successive approximation), a standard technique in functional analysis, is used to find an approximate closed form solution of this integral equation. Suppose an integral equation has a form of

$$Tf(x) = g(x) + \lambda \int_B K(x,s)Gf(s)\,\mathrm{d}x, \tag{19}$$

where $f(x)$ is the desired solution, K and g are given functions, T and G are given operators.

By assuming an initial function f_1, T leads to a new function of f_2, or in general,

$$f_{n+1} = Tf_n. \tag{20}$$

This iteration method is expected to converge rapidly. The error caused in the iteration may be estimated by the following equation

$$\|f_n - f\| \leqslant \varepsilon^n, \tag{21}$$

where f is the true solution, f_n, the estimation after the nth iteration, and ε, a positive number much smaller than 1. $\|f_n - f\|$ denotes the maximum type norm on a suitable space of functions containing the functional sequence $\{f_n\}$ (Wang and Hsu, 1994). More details are provided in appendix.

In the following analysis, a linearly descending cohesion versus crack opening displacement relation is used to simulate the softening properties of the concrete material. The cohesion versus crack opening displacement relation reads

$$\frac{\sigma}{f_\mathrm{t}} = 1 - \frac{\delta}{w_\mathrm{c}} \tag{22}$$

in which, σ is the cohesion in accordance with the crack opening displacement δ in the plastic zone, w_c, the critical crack opening displacement of the concrete, and f_t, the tensile stress of the concrete.

In the case that the crack opening displacement in the plastic zone is given by a function of the abscissa, or $\delta(b)$, the local cohesion can be uniquely determined using Eq. (22). However, the crack opening displacement $\delta(b)$ is unknown and shall be determined by the governing equation.

In order to find the approximate closed form solution, the iterative method may be performed by assuming an initial deformed shape of the plastic zone. For simplicity, a linear function is assumed for the first iteration. Thus, the cohesion in the plastic zone may be found as follows:

$$\frac{\sigma}{f_t} = 1 - \frac{\delta(a)}{w_c}\frac{c-b}{c-a}, \tag{23}$$

where a is the initial crack length, b, the integral variable, and c, the sum of the initial crack length and the size of the plastic zone, respectively. $\delta(a)$ is the crack tip opening displacement.

By doing so, the integral equation becomes an integration of a given function. Introducing dimensionless parameters: $\alpha = a/c,\ s = x/c,\ z = \xi/c,\ \tau = b/c$ and $k^* = \delta(a)/w_c(1-\alpha)$. Eq. (24) becomes,

$$\delta(s) = \frac{4\sigma_0}{E}c\sqrt{1^2 - s^2} - \frac{8f_t c}{\pi E}\int_s^1 \frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\zeta^2 - s^2}}\int_\alpha^\zeta \frac{1 - k^*(1-\tau)}{\sqrt{\zeta^2 - \tau^2}}\,\mathrm{d}\tau. \tag{24}$$

Eq. (24) leads to the following formula for crack opening displacement:

$$\begin{aligned}\delta(s) = {}& \sqrt{1-s^2}\left[(1-k^*)\left(\frac{\pi}{2} - \arcsin\alpha\right) + k^*\sqrt{1-\alpha^2}\right] \\ & + \alpha(1-k^*)\int_s^1 \sqrt{\frac{\zeta^2 - s^2}{\zeta^2 - \alpha^2}}\frac{1}{\zeta}\,\mathrm{d}\zeta - \frac{k^*}{2}\sqrt{1-\alpha^2}\sqrt{1-s^2} + \frac{k^*}{2}\left\{\frac{s^2-\alpha^2}{2}\ln\left[\frac{\sqrt{1-s^2} - \sqrt{1-\alpha^2}}{s^2-\alpha^2}\right]\right\},\end{aligned} \tag{25}$$

and the boundary condition yields

$$(1-k^*)\left(\frac{\pi}{2} - \arcsin\alpha\right) + k^*\sqrt{1-\alpha^2} = \frac{\pi}{2}\frac{\sigma_0}{f_t}. \tag{26}$$

Thus, the solution may be simplified as follows:

$$\begin{aligned}\delta(s) = {}& \frac{\pi}{2}\frac{\sigma_0}{f_t}\sqrt{1-s^2} + \alpha(1-k^*)\int_s^1 \sqrt{\frac{\zeta^2 - s^2}{\zeta^2-\alpha^2}}\frac{1}{\zeta}\,\mathrm{d}\zeta - \frac{k^*}{2}\sqrt{1-\alpha^2}\sqrt{1-s^2} \\ & + \frac{k^*}{2}\left\{\frac{s^2-\alpha^2}{2}\ln\left[\frac{\sqrt{1-s^2}-\sqrt{1-\alpha^2}}{s^2-\alpha^2}\right]\right\}.\end{aligned} \tag{27}$$

The crack tip opening displacement may be found by letting $s \to \alpha$, and it reads

$$\frac{\delta(\alpha)}{w_c} = \frac{8f_t}{\pi E}\frac{c}{w_c}\left(\frac{\pi}{2}\frac{\sigma_0}{f_t}\sqrt{1-\alpha^2} - \frac{k^*}{2}(1-\alpha^2)\right) = \frac{\Psi}{\alpha}\left(\frac{\pi}{2}\frac{\sigma_0}{f_t}\sqrt{1-\alpha^2} - \frac{k^*}{2}(1-\alpha^2)\right), \tag{28}$$

where

$$\Psi = \frac{8f_t a}{w_c \pi E}, \tag{29}$$

and

$$k^* = \frac{\ln(1/\alpha)}{\frac{1-\alpha}{\Psi} + \ln(1/\alpha) + \frac{1-\alpha^2}{\alpha}}. \tag{30}$$

This functional iterative method is expected to have fast convergence. With a properly chosen initial deformed shape, the first iteration may give a very good approximation of the true solution.

For a centrally cracked plate, an unstable crack progress occurs when the crack tip opening displacement reaches the critical value of w_c under the applied stress σ_0. The maximum normalized tensile strength, σ_0/f_t of the specimens with different crack lengths predicted by the above equations are listed in Table 1. As the crack length increases from 0 to 10 m, the predicted nominal stress at failure of the specimen drops from $1.0f_t$ to $0.151f_t$. Calculation shown in Table 1 assumes a concrete mixture with a tensile strength of 2.9 MPa, a critical crack opening displacement of 120 μm, a modulus of elasticity of 29.6 MPa. A 10-m crack may not seem practical in building or bridge structures, but such cracks are not rare in mass concrete structures, such as gravity dams.

By the classical plastic zone theory, the critical J integral value is $1/2w_c f_t$, or 174 N/m for the material chosen. In the case of small scale yielding, the critical stress intensity factor is equal to $\sqrt{J_c E}$ or 2.269×10^6 N/m$^{-3/2}$.

The result from the generalized plastic zone theory is given in Table 1. The numerical result plotted in a logarithmic scale graph is shown in Fig. 5. The graph shows the typical 1:2 slope in the linear elastic range. It also shows the smooth transition from the strength failure criterion to the stress intensity factor criterion of the material. Therefore, considering the fact that results listed in Table 1 and Fig. 5 are computed from the first iterative solution, it clearly demonstrates the effectiveness and theoretical significance of the proposed general formulation. This result also reflects the fast convergence of the iterative method used in this analysis.

Eq. (29) may be written in the form of stress intensity factor and J_c as follows:

$$\Psi = \frac{4\pi a f_t^2}{\frac{1}{2} w_c f_t \pi^2 E} = \frac{4K_{nf}^2}{\pi^2 J_c E} \approx 0.4\frac{K_{nf}^2}{J_c E}, \tag{31}$$

where K_{nf} is a nominal stress intensity factor of structures with crack length $= a$ under maximum stress allowed under plastic yielding criterion. For instance, K_{nf} for a centrally cracked wide plate with tensile strength of f_t is $\sqrt{\pi a} f_t$. Defining a brittleness index ψ,

Table 1
Nominal stress at failure predicted by generalized plastic zone theory

a(m)	σ_0/f_t	Ψ	ψ	a/c	K (N/m$^{-3/2}$)
0.001	0.995	0.002	0.005	0.003	1.618×10^5
0.025	0.929	0.052	0.128	0.074	7.547×10^5
0.050	0.881	0.104	0.257	0.137	1.012×10^6
0.100	0.809	0.208	0.513	0.241	1.315×10^6
0.200	0.711	0.416	1.026	0.391	1.635×10^6
0.300	0.644	0.624	1.540	0.494	1.813×10^6
0.400	0.593	0.832	2.053	0.568	1.928×10^6
0.500	0.553	1.040	2.566	0.623	2.008×10^6
0.600	0.520	1.247	3.077	0.666	2.069×10^6
0.700	0.492	1.455	3.590	0.700	2.115×10^6
0.800	0.468	1.663	4.103	0.728	2.152×10^6
0.900	0.448	1.871	4.617	0.751	2.183×10^6
1.000	0.430	2.079	5.130	0.771	2.208×10^6
5.000	0.211	10.395	25.649	0.945	2.422×10^6
10.000	0.151	20.791	51.300	0.972	2.453×10^6

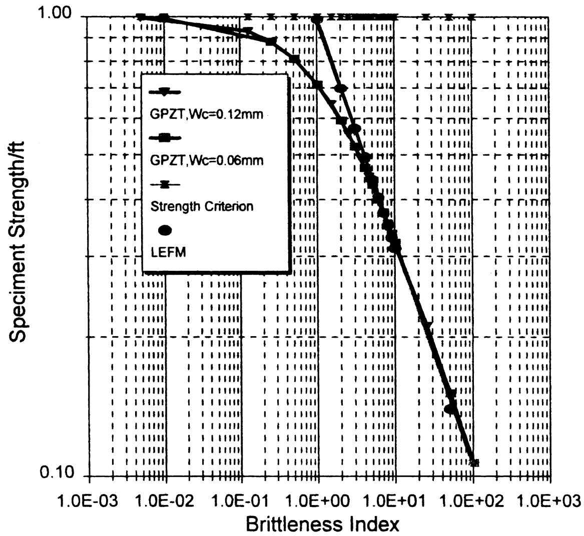

Fig. 5. Size effect for cemmentitious material by generalized plastic zone theory.

$$\psi = \frac{K_{\rm nf}^2}{J_{\rm c}E}. \tag{32}$$

Experimental results have indicated that the nominal stress at failure of concrete structures decreases with increasing in structural size. The traditional explanation was based on Weibull's weakest link statistical theory. (Weibull, 1939). This phenomenon is believed to be related to crack propagation, and thus may be better explained by using the theory of fracture mechanics. But the difficulty is to achieve a smooth transition between strength criterion and fracture energy criterion.

Since the proposed brittleness index is structural size related factor, the size effect may be represented by using the brittleness factor. Fig. 5 shows the relation between the brittleness index and nominal stress at failure of centrally cracked wide concrete plates. Structures with smaller brittleness indexes have higher ductility, and the nominal failure stresses are close to that predicted by the plastic theory. These structures have brittleness indexes less than 0.05 as indicated in Fig. 5. Note that $\Psi = 8f_{\rm t}a/w_{\rm c}\pi E$ for a centrally cracked wide plate, Ψ also reflect the ratio of $a/w_{\rm c}$, where $w_{\rm c}$ is the critical crack opening width.

Structures with brittleness indexes greater than 5 may be classified as brittle structures. Linear elastic fracture mechanics may be used to analyze these structures. The failure criterion in this case is $K \leqslant K_{\rm IC}$.

However, for the structures with a brittleness index between 0.05 and 5, neither the traditional theory of plasticity nor linear fracture mechanics is able to predict the structural behavior with higher accuracy. The use of non-linear fracture mechanics is desirable. If the strip-yielding model is to be used, the proposed method may provide effective tool for the materials with variable cohesion.

The maximum sizes of plastic zones at the time of failure are listed in Table 1 and they are illustrated in Figs. 6 and 7. Fig. 7 shows a linear relationship between the maximum size of the plastic zone and the brittleness index in a logarithmic scale. Fig. 8 shows the maximum stress intensity factor of the specimen at the time of failure.

The brittleness index ψ has a straightforward physical meaning. It is the ratio between the applied fracture energy-release rate and the fracture energy toughness ($J_{\rm c}$) at the time of failure of an ideal plastic-yielding-brittle-fracture material (Wang, 1995). The definition of an ideal plastic-yielding-brittle-fracture material is the material that will fail due to either plastic flow or unstable crack propagation. Thus, if $\psi < 1$, a structure will fail due to plastic flow, or brittle fracture, i.e.:

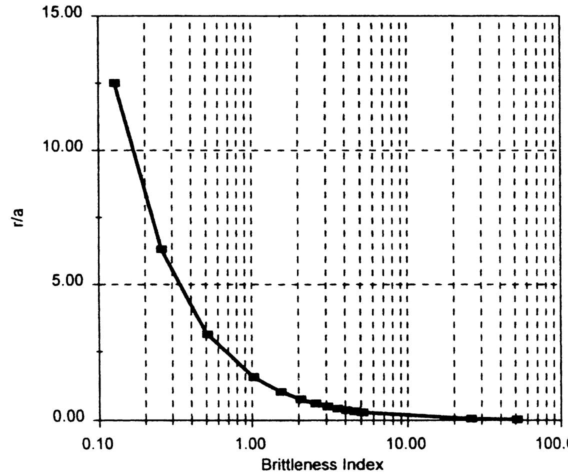

Fig. 6. Normalized maximum plastic zone size at the time of failure (of normal scale).

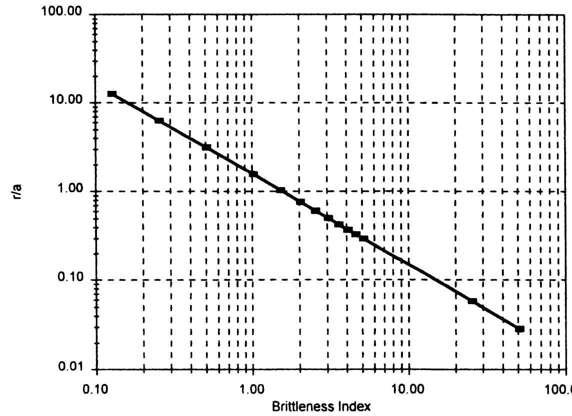

Fig. 7. Normalized maximum plastic zone size at the time of failure (of logarithmic scale).

$$\sigma = \sigma_y \quad \text{for } \psi < 1,$$
$$K = K_{\mathrm{IC}} \quad \text{for } \psi \geqslant 1.$$

These two criteria are straight lines in a logarithm scaled coordinate system as shown in Fig. 5. The line indicates the fracture criterion has a slope of 1:2 as expected. Engineering materials must converge to these lines in extreme cases when $\psi \ll 1$ and $\psi \gg 1$. Theoretically, the theory of plasticity may only apply to the structures with $\psi = 0$. Also, the principle of linear fracture mechanics is true only when $\psi = \infty$. However, in engineering practice, the theory of plasticity and linear fracture mechanics may apply to the structures of certain ψ value with considerable accuracy. Better theories should be able to predict a nominal stress that converges to result from both the plastic theory and the linear fracture mechanics. They should also provide a smooth transition between the two.

Results from the analysis of another material with a smaller critical crack opening displacement of 0.06 mm is also shown in Fig. 5. The results from these two analyses are almost identical as illustrated in Fig. 5,

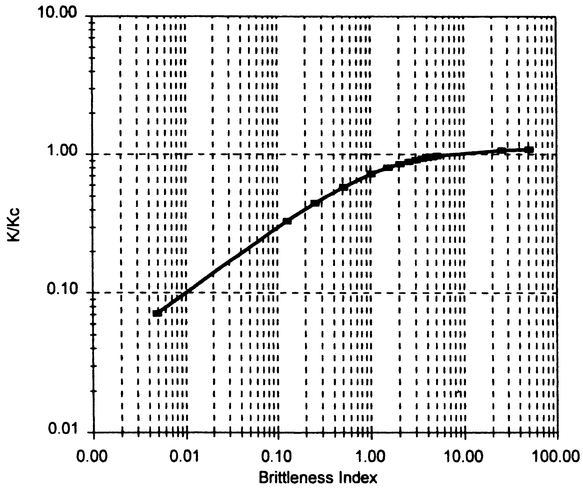

Fig. 8. Maximum stress intensity factor at the time of failure.

even though their critical crack tip opening displacements differ from one another by 100%. This observation suggests that this brittleness index is independent of the material and structural properties, and may be used to identify the ductility and failure mode of a structure. Further studies are necessary to verify this theorem.

4. Conclusions

The findings of this research can be summarized in the following:

1. The general formulation for strip yielding model presented in this article is able to predict the non-linear fracture properties of materials with constant or variable cohesion. When applying to softening materials, this method is able to predict the size effect. The nominal stress at failure predicted by the proposed method converges well to results by the plastic theory and the linear elastic fracture mechanics. The proposed method also provides a smooth transition between the two.
2. A brittleness index derived based on the results from the generalized plastic zone theory may be used as an indicator to determine the ductility of structures. The classical theory of plasticity may be used for structures with a brittleness index less than 0.05. When this index is greater than 5, linear elastic fracture mechanics becomes applicable. However, structures with a brittleness index between 0.05 and 5 shall be analyzed using non-linear fracture mechanics.

Appendix A. Convergence proof for integral equations

The integral equation

$$\delta(x) = \frac{4\sigma_0}{E'}\sqrt{c^2 - x^2} - \frac{8}{\pi E'}\int_x^c \frac{\xi\,\mathrm{d}\xi}{\sqrt{\xi^2 - x^2}}\int_a^x \frac{\sigma[\delta(s)]}{\sqrt{\xi^2 - s^2}}\,\mathrm{d}s \tag{A.1}$$

represents a free-boundary problem since c needs to be determined. We use the auxillary equation

$$\frac{\pi\sigma_0}{2} = \int_a^c \frac{\sigma[\delta(s)]}{\sqrt{c^2 - s^2}}\,\mathrm{d}s \tag{A.2}$$

in order to determine c. Assuming that δ is continuous on the interval $[a, c]$, it is not difficult to show that Eq. (A.2) is a necessary and sufficient condition for $\delta(x) = o(\sqrt{c^2 - x^2})$ as $x \to c-$; i.e.,

$$\frac{\delta(x)}{\sqrt{c^2 - x^2}} \to 0 \quad \text{as } x \to c \text{ with } x < c.$$

To see this note that for x near c Eq. (A.1) can be written in the form

$$\delta(x) = \frac{4\sigma_0}{E'}\sqrt{c^2 - x^2} - \frac{8}{\pi E'}\int_x^c \frac{\xi}{\sqrt{\xi^2 - x^2}}\left[\int_a^c \frac{\sigma[\delta(s)]\,\mathrm{d}s}{\sqrt{c^2 - s^2}} + \theta\right]\mathrm{d}\xi,$$

where $\theta \to 0$ uniformly as $x \to c-$. Then upon substitution of Eq. (A.2), we obtain

$$\delta(x) = -\frac{8}{\pi E'}\int_x^c \frac{\theta\xi\,\mathrm{d}\xi}{\sqrt{\xi^2 - x^2}}.$$

For the convergence proof, we assume that σ is a non-negative, bounded function with continuous derivatives of all orders. Moreover, $\sigma(\delta)$ is assumed to satisfy

$$\|\sigma(\tilde{\delta}) - \sigma(\delta)\| \leqslant k_1\|\tilde{\delta} - \delta\|, \tag{A.3}$$

where k_1 is a positive constant much smaller than E' and $\|\|$ is the uniform norm for continuous functions defined by

$$\|f\| \leqslant \max\{|f(x)| : a \leqslant x \leqslant c\}.$$

Note that the above assumptions are consistent with the application at hand.

It is easy to show that the assumptions imply that $I(\delta,c)$, defined to be the integral in Eq. (A.2), is a monotone increasing function of c for a fixed continuous δ. Consequently, Eq. (A.2) has a unique solution of the form $c = c(\delta)$ satisfying

$$I[\delta, c(\delta)] = \frac{\pi\sigma_0}{2}$$

It is now easy to show that the above assumptions imply that there is a positive k_2 much smaller than E' such that

$$\|c(\tilde{\delta}) - c(\delta)\| \leqslant k_2\|\tilde{\delta} - \delta\|. \tag{A.4}$$

With the above the notation, Eq. (A.1) can be rewritten in the compact functional form

$$\delta = T(\delta), \tag{A.5}$$

where

$$T(\delta)(x) := \frac{4\sigma_0}{E'}\sqrt{c(\delta)^2 - x^2} - \frac{8}{\pi E'}\int_x^{c(\delta)} \frac{\xi\,\mathrm{d}\xi}{\sqrt{\xi^2 - x^2}}\int_a^x \frac{\sigma[\delta(s)]}{\sqrt{\xi^2 - s^2}}\,\mathrm{d}s.$$

On using Eq. (A.3) and Eq. (A.4) in Eq. (A.5) after a simple computation of estimates, we find that

$$\|T(\tilde{\delta}) - T(\delta)\| \leqslant k\|\tilde{\delta} - \delta\|, \tag{A.6}$$

where k is a positive number much smaller than 1. Hence T is a contracting mapping, so the successive approximations defined by

$$\delta_{n+1} = T(\delta_n), \quad n \geqslant 1$$

converge very rapidly to a continuous solution δ_*. In fact, we have

$$\|\delta_{n+1} - \delta_*\| \leqslant \frac{k^n}{1-k}\|\delta_2 - \delta_1\|$$

(cf. J. Dugundji, 1966). The corresponding value of the end point c is then obtained by setting $c = c(\delta_*)$.

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